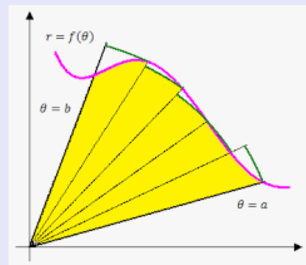
Calculus II

Lecture 15



Feb 19-8:47 AM

Class QZ 11 (15 pts)

Find the exact value of the Surface area if the arc length given by $f(x)=\frac{x^4}{16}+\frac{1}{2x^2}$ for $1\le x\le 2$ rotated about Y-axis.

$f'(x)=\frac{4x^3}{16}-\frac{1}{x^3}=\boxed{\frac{x^3}{4}-\frac{1}{x^3}}$ $\quad [f'(x)]^2=\boxed{\frac{x^6}{16}-\frac{1}{2}+\frac{1}{x^6}}$

$1+[f'(x)]^2=\frac{x^6}{16}+\frac{1}{2}+\frac{1}{x^6}=\boxed{\left(\frac{x^3}{4}+\frac{1}{x^3}\right)^2}$

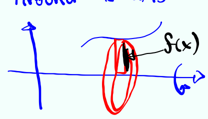
$\sqrt{1+[f'(x)]^2}=\boxed{\frac{x^3}{4}+\frac{1}{x^3}}$

$$S=\int_1^2 2\pi x\left(\frac{x^3}{4}+\frac{1}{x^3}\right)dx=2\pi\int_1^2\left[\frac{x^4}{4}+\frac{1}{x^2}\right]dx$$

$$=2\pi\left[\frac{x^5}{20}-\frac{1}{x}\right]\Big|_1^2=2\pi\left[\left(\frac{32}{20}-\frac{1}{2}\right)-\left(\frac{1}{20}-1\right)\right]=2\pi\left[\frac{31}{20}+\frac{1}{2}\right]$$

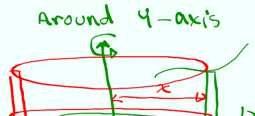
$$=2\pi\cdot\frac{41}{20}=\boxed{\frac{41\pi}{10}}$$

Around x-axis



$$S=\int_a^b 2\pi f(x)\sqrt{1+[f'(x)]^2}\,dx$$

Around y-axis

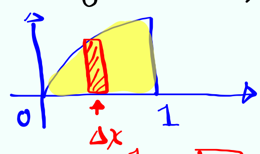


$$S=\int_a^b 2\pi x\sqrt{1+[f'(x)]^2}\,dx$$

Jul 3-10:54 AM

Class QZ 12

Consider the region enclosed by $f(x)=\sqrt{x}$, $g(x)=0$, and $x=1$



1) Draw & find its Area.

$$A=\int_0^1 \sqrt{x}\,dx=\frac{x^{3/2}}{3/2}\Big|_0^1=\frac{2}{3}x\sqrt{x}\Big|_0^1=\boxed{\frac{2}{3}}$$

2) Find its Centroid.

$$\bar{x}=\frac{1}{A}\int_a^b x[f(x)-g(x)]dx=\frac{3}{2}\int_0^1 x\sqrt{x}\,dx=\frac{3}{2}\int_0^1 x^{3/2}dx$$

$$=\frac{3}{2}\cdot\frac{x^{5/2}}{5/2}\Big|_0^1=\boxed{\frac{3}{5}}$$

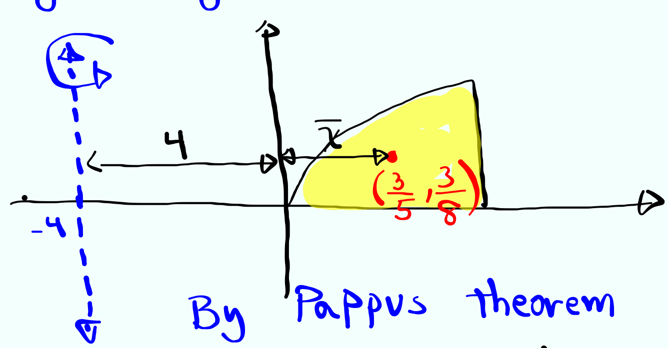
$$\bar{y}=\frac{1}{A}\int_a^b \frac{1}{2}\left[(f(x))^2-(g(x))^2\right]dx$$

$$=\frac{3}{2}\int_0^1 \frac{1}{2}(\sqrt{x})^2dx=\frac{3}{4}\int_0^1 x\,dx=\frac{3}{4}\cdot\frac{x^2}{2}\Big|_0^1=\boxed{\frac{3}{8}}$$

Centroid $\left(\frac{3}{5},\frac{3}{8}\right)$

Jul 8-6:57 AM

Rotate that region by $x=-4$, find its Volume.



By Pappus theorem

$$V=A\cdot 2\pi\cdot(\bar{x}+4)$$

$$=\frac{2}{3}\cdot 2\pi\cdot\left(\frac{3}{5}+4\right)=\boxed{}$$

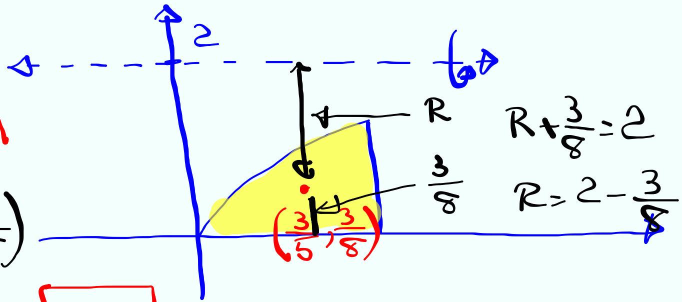
Jul 8-8:23 AM

Now rotate the region by $y=2$, find its Volume

By Pappus thrm

$$V = A \cdot 2\pi \cdot \left(2-\frac{3}{8}\right)$$

$$= \frac{\cancel{2}}{3}\cdot \cancel{2}\pi \cdot \frac{13}{\cancel{8}\,2} = \boxed{\frac{13\pi}{6}}$$



$R+\frac{3}{8}=2$

$R=2-\frac{3}{8}$

Jul 8-8:27 AM

Parametric Equations

$x=f(t)$

$y=g(t)$

$a\le t\le b$

Initial Point
$(f(a),\ g(a))$

Terminal Point
$(f(b),\ g(b))$

$x=\sqrt{t}$

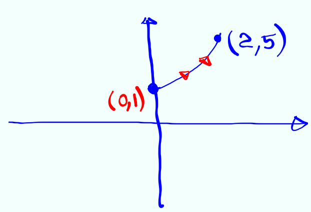
$y=t+1$

$0\le t\le 4$

$x=\sqrt{t} \rightarrow x^2=(\sqrt{t})^2 \rightarrow x^2=t$

$y=t+1 \rightarrow y=x^2+1$

Parabola



I.P.

$x=f(0)=\sqrt{0}$

$y=g(0)=0+1$ $\rightarrow (0,1)$

T.P.

$x=f(4)=\sqrt{4}=2$

$y=g(4)=4+1=5$ $\rightarrow (2,5)$

Jul 8-8:31 AM

$x = 2\cos t$

$y = 2\sin t$

$0 \le t \le 2\pi$

$\cos t = \frac{x}{2}$

$\sin t = \frac{y}{2}$

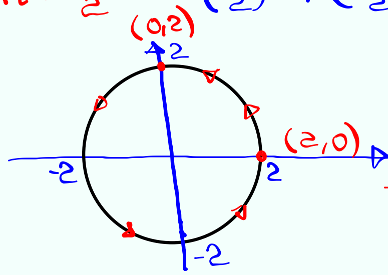
$\cos^2 t + \sin^2 t = 1$

$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{2}\right)^2 = 1 \rightarrow x^2 + y^2 = 4$

Circle

center $(0,0)$

Radius 2



$t = 0$

$x = 2\cos 0 = 2$

$y = 2\sin 0 = 0$

$t = \frac{\pi}{2}$

$x = 2\cos\frac{\pi}{2} = 0$

$y = 2\sin\frac{\pi}{2} = 2$

Jul 8-8:37 AM

$x = 2 + 3\cos t$

$y = -1 + 4\sin t$

$0 \le t \le \pi$

$\cos t = \frac{x-2}{3}$

$\sin t = \frac{y+1}{4}$

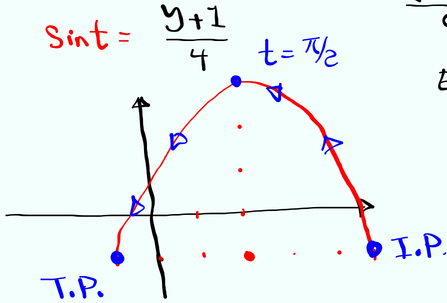
use $\cos^2 t + \sin^2 t = 1$ identity

$\frac{(x-2)^2}{9} + \frac{(y+1)^2}{16} = 1$

Ellipse

Center $(2,-1)$

$a = 3 \quad b = 4$



I.P. $t = 0$

$x = 2 + 3\cos 0 = 5$

$y = -1 + 4\sin 0 = -1$

T.P. $t = \pi$

$x = 2 + 3\cos\pi = -1$

$y = -1 + 4\sin\pi = -1$

$t = \frac{\pi}{2}$

$x = 2 + 3\cos\frac{\pi}{2} = 2$

$y = -1 + 4\sin\pi/2 = 3$

Jul 8-8:42 AM

$x = \sinh t$

$y = \cosh t$

1) Eliminate t, find equation in x & y only.

2) Graph it.

Tip:

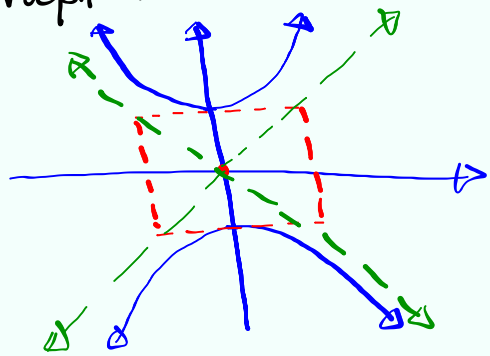
$\cosh^2 x - \sinh^2 x = 1$

$y^2 - x^2 = 1$

Hyperbola

cent $(0,0)$

$a=1$, $b=1$



Jul 8-8:51 AM

$x = e^{2t}$

$y = t + 1$

Eliminate t

graph the curve

Discuss Domain & Range

$x = e^{2t}$

$\ln x = \ln e^{2t}$

$\ln x = 2t \cancel{\ln e}^{\,1}$

$\frac{1}{2}\ln x = t$

$y = \frac{1}{2}\ln x + 1$

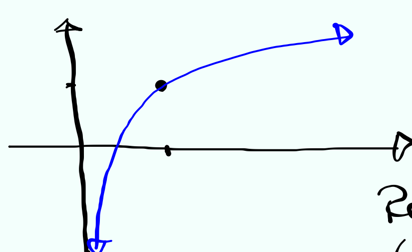
$x = 1 \quad y = 1$

$e^{2t} > 0$

$x > 0$

Domain

$(0, \infty)$



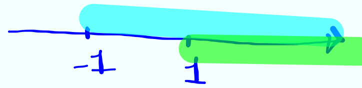
Range

$(-\infty, \infty)$

Jul 8-8:57 AM

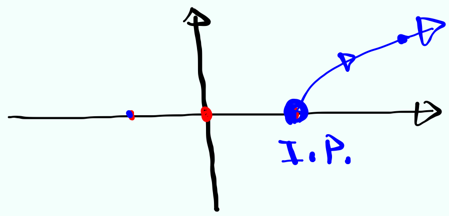
$x = \sqrt{t+1}$ $\quad t \geq 1$ $\quad t+1 \geq 0 \quad t \geq -1$

$y = \sqrt{t-1}$ $\quad t-1 \geq 0 \quad t \geq 1$



$x^2 = t+1 \qquad x^2 - 1 = t \qquad x^2 - 1 = y^2 + 1$

$y^2 = t - 1 \qquad y^2 + 1 = t \qquad x^2 - y^2 = 2$



Hyperbola

open sideways

center at (0, 0)

$t = 1 \rightarrow$ I.P. $\quad x^2 = 2 \quad x = \sqrt{2}, \; y = 0 \qquad t = 5 \quad x = \sqrt{6}, \; y = 2$

Jul 8-9:04 AM

First Derivative

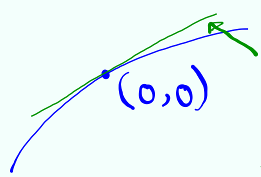
$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$x = t^2 - 2t$

$y = t + 1$

$$\frac{dy}{dx} = \frac{1}{2t-2}$$

Find the equation of tan. line to the graph of $x = 6\sin t$, $y = t^2 + t$ at $t = 0$.



$$m = \left.\frac{dy}{dx}\right|_{(0,0)}$$

$$m = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \left.\frac{2t+1}{6\cos t}\right|_{t=0} = \frac{2(0)+1}{6\cos 0} = \frac{1}{6}$$

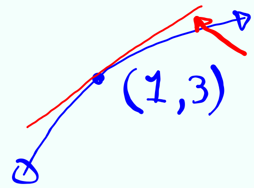
$$y - 0 = \frac{1}{6}(x - 0)$$

$$\boxed{y = \frac{1}{6}x}$$

Jul 8-9:11 AM

$x = 1 + \ln t$

$y = t^2 + 2$



$m = \left.\frac{dy}{dx}\right|_{(1,3)}$

find eqn of the normal line to the curve at $t = 1$.

$m = \dfrac{-1}{\left.\frac{dy}{dx}\right|_{(1,3)}}$ $t=1$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t}{\frac{1}{t}} = 2t^2\Big|_{t=1} = 2$$

$$y - 3 = \frac{-1}{2}(x - 1) \Rightarrow \boxed{y =}$$

Jul 8-9:18 AM

Second Derivative

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{\frac{dx}{dt}}$$

$x = t^2 + 1$

$y = t^2 + t$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t+1}{2t}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left[\frac{2t+1}{2t}\right]}{2t}$$

$$= \frac{\frac{d}{dt}\left[1 + \frac{1}{2}t^{-1}\right]}{2t}$$

$$= \frac{0 + \frac{1}{2} \cdot \frac{-1}{t^2}}{2t} = \frac{\frac{-1}{2t^2}}{2t}$$

$$\boxed{\frac{d^2y}{dx^2} = \frac{-1}{4t^3}}$$

$t > 0$ $y'' < 0$ C.D.

$t < 0$ $y'' > 0$ C.U.

Jul 8-9:26 AM

$x = e^t$ Find $\frac{dy}{dx}$ & $\frac{d^2y}{dx^2}$

$y = \frac{t}{e^t}$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{1\cdot e^t - t\cdot e^t}{(e^t)^2}}{e^t} = \frac{\frac{e^t(1-t)}{e^{2t}}}{e^t} = \frac{e^t(1-t)}{e^{3t}} = \frac{1-t}{e^{2t}}$$

$1 - t = 0 \quad t = 1$

$f' > 0$ $f' < 0$

Inc. $t=1$ Dec.

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left[\frac{1-t}{e^{2t}}\right]}{e^t} = \frac{\frac{-1\cdot e^{2t} - (1-t)\cdot 2e^{2t}}{e^{4t}}}{e^t} = \frac{e^{2t}(-1-2+2t)}{e^{5t}}$$

$$\frac{d^2y}{dx^2} = \frac{2t-3}{e^{3t}}$$

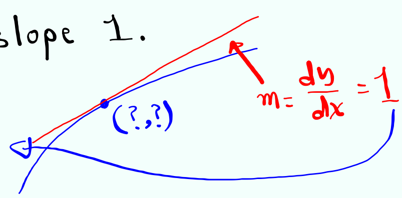
$f'' < 0$ $f'' > 0$

C.D. $t = \frac{3}{2}$ C.U.

Jul 8-9:34 AM

Find the point on the graph of $x = 2t^3$ and $y = 1 + 4t - t^2$ where tan. line has slope 1.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{4-2t}{6t^2}$$



$$\frac{4-2t}{6t^2} = 1$$

$6t^2 = 4 - 2t$

$3t^2 = 2 - t$

$3t^2 + t - 2 = 0$

$(3t-2)(t+1) = 0$

$t = \frac{2}{3} \quad t = -1$

Two points

$t = \frac{2}{3} \rightarrow x =$

$y =$

$t = -1 \rightarrow x =$

$y =$

Jul 8-10:00 AM

$x = \sqrt{t}$ $\quad \frac{dx}{dt} = \frac{1}{2\sqrt{t}}$ $\quad dx = \frac{1}{2\sqrt{t}}\,dt$

$y = t^2 - 2t$

Find the area enclosed by this graph and x-axis

on the x-axis $y = 0$

$t^2 - 2t = 0$

$t = 0\ ,\ t = 2$

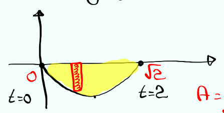
$t = 0 \to x = 0,\ y = 0 \to (0,0)$

$t = 2 \to x = \sqrt{2},\ y = 0 \to (\sqrt{2}, 0)$

what $t = 1$?

$x = \sqrt{1} = 1$

$y = 1^2 - 2(1) = -1$



$A = \int_0^{\sqrt{2}} [0 - y]\,dx$

$= \int_0^2 [0 - (t^2 - 2t)] \cdot \frac{1}{2\sqrt{t}}\,dt$

$= -\frac{1}{2}\int_0^2 \left(\frac{t^2}{\sqrt{t}} - \frac{2t}{\sqrt{t}}\right)dt = -\frac{1}{2}\left[\int_0^2 (t^{3/2} - 2\sqrt{t})\,dt\right]$

$= -\frac{1}{2}\left[\frac{t^{5/2}}{5/2} - 2\cdot\frac{t^{3/2}}{3/2}\right]\Big|_0^2$

$= -\frac{1}{2}\left[\frac{2}{5}t^2\sqrt{t} - \frac{4}{3}t\sqrt{t}\right]\Big|_0^2$

$= -\frac{1}{2}\left[\frac{2}{5}\cdot 4\sqrt{2} - \frac{4}{3}2\sqrt{2}\right] =$

$= -\frac{1}{2}\cdot 8\sqrt{2}\left[\frac{1}{5} - \frac{1}{3}\right]$

$= \frac{-1}{2} 8\sqrt{2} \cdot \frac{-2}{15} = \boxed{\frac{8\sqrt{2}}{15}}$

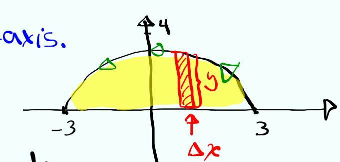
Jul 8-10:07 AM

$x = 3\cos\theta$ $\quad \left(\frac{x}{3}\right)^2 + \left(\frac{y}{4}\right)^2 = 1$

$y = 4\sin\theta$ $\quad \frac{x^2}{9} + \frac{y^2}{16} = 1$

$0 \le \theta \le \pi$ $\quad a = 3 \quad b = 4$

Find the area below this graph and above the x-axis.



$A = \int_{-3}^{3} y\,dx$

$= \int_{\pi}^{0} 4\sin\theta \cdot -3\sin\theta\,d\theta$

$= 12\int_0^{\pi} \sin^2\theta\,d\theta$

$= 12\int_0^{\pi} \frac{1 - \cos 2\theta}{2}\,d\theta = 6\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\pi}$

$= 6\cdot\pi = \boxed{6\pi}$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Area $= ab\pi$

Top half $\frac{ab\pi}{2}$

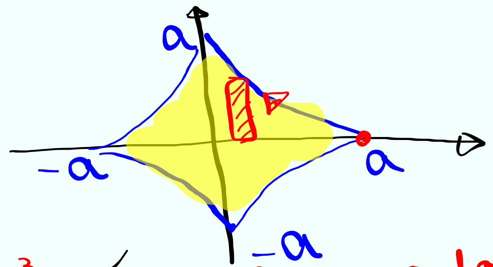
Our Problem $\frac{3\cdot 4\pi}{2} = 6\pi$

Jul 8-10:20 AM

Find the shaded area below where

$x = a\cos^3\theta$

$y = a\sin^3\theta$



$$A = 4\int_0^a y\,dx = 4\int_{\pi/2}^{0} a\sin^3\theta \cdot -3a\cos^2\theta \cdot \sin\theta\,d\theta$$

$$= 12a^2\int_0^{\pi/2} \sin^4\theta \cdot \cos^2\theta\,d\theta = 12a^2 \boxed{}$$

Jul 8-10:36 AM

$x = 1 + 3t^2$

$y = 4 + 2t^3$

$0 \le t \le 1$

Find the arc length.

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx$$

$$f'(x) = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{6t^2}{6t} = t$$

$$L = \int_0^1 \sqrt{1 + t^2}\;6t\,dt$$

$u = 1 + t^2$

$du = 2t\,dt$

$$= \int_1^2 \sqrt{u}\cdot 3\,du \qquad = \boxed{}$$

Jul 8-10:44 AM

Here is a new arc length formula

$$L=\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

$$=\int \sqrt{\left(\frac{dx}{dt}\right)^2\left[1+\frac{\left(\frac{dy}{dt}\right)^2}{\left(\frac{dx}{dt}\right)^2}\right]}\,dt$$

$$=\int \frac{dx}{\cancel{dt}}\sqrt{1+\left[\frac{dy}{dx}\right]^2}\,\cancel{dt}$$

$$=\int \sqrt{1+[f'(x)]^2}\,dx$$

Jul 8-10:49 AM

Use $L=\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$ to find the arc length of $x=1+3t^2$, $y=4+2t^3$ for $0\le t\le 1$.

$$L=\int_0^1 \sqrt{(6t)^2+(6t^2)^2}\,dt$$

$$=\int_0^1 \sqrt{36t^2+36t^4}\,dt$$

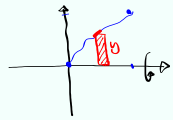
$$=\int_0^1 6t\sqrt{1+t^2}\,dt$$

Jul 8-10:53 AM

$x = t^3$ Find the surface area

$y = t^2$ by rotating the given graph

$0 \le t \le 1$ by x-axis.



$$S = \int_0^1 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_0^1 2\pi t^2 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^1 2\pi t^2 \sqrt{(3t^2)^2 + (2t)^2}\, dt$$

$$9t^4 + 4t^2$$

$$= 2\pi \int_0^1 t^2 \sqrt{9t^4 + 4t^2}\, dt = 2\pi \int_0^1 t^2 \cdot t \sqrt{9t^2 + 4}\, dt$$

$u = 9t^2 + 4$

$du = 18t\, dt$

$\frac{du}{18} = t\, dt$

$$= 2\pi \int_4^{13} \frac{u-4}{9} \cdot \sqrt{u}\, \frac{du}{18}$$

$$= \frac{\pi}{81} \int_4^{13} (u\sqrt{u} - 4\sqrt{u})\, du$$

$$= \frac{\pi}{81} \left[\frac{u^{5/2}}{5/2} - \frac{4u^{3/2}}{3/2} \right]_4^{13}$$

$$= \frac{\pi}{81} \left[\frac{2}{5} u^2 \sqrt{u} - \frac{8}{3} u\sqrt{u} \right]_4^{13}$$

$$\boxed{\frac{2\pi}{1215}\left(247\sqrt{13} + 64\right)}$$

$$= \frac{\pi}{81} \left[\frac{2}{5} \cdot 13^2 \sqrt{13} - \frac{8}{3} 13\sqrt{13} - \frac{2}{5} \cdot 4^2 \sqrt{4} + \frac{8}{3} \cdot 4\sqrt{4} \right]$$

$$= \frac{\pi}{81} \left[\frac{338\sqrt{13}}{5} - \frac{104\sqrt{13}}{3} - \frac{64}{5} + \frac{64}{3} \right]$$

$$= \frac{\pi}{81} \left[\frac{1014\sqrt{13} - 520\sqrt{13} - 192 + 320}{15} \right] = \frac{\pi}{1215} \left[494\sqrt{13} + 128 \right]$$

$$= \frac{\pi}{1215} \left[2(247\sqrt{13} + 64) \right]$$

$$= \boxed{\frac{2\pi}{1215} \left[247\sqrt{13} + 64 \right]}$$

Jul 8-10:56 AM